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A theoretical model for the shock stand-off distance in frozen and equilibrium flows

By H. OLIVIER

Shock Wave Laboratory, RWTH Aachen, Templergraben 55, 52056 Aachen, Germany

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In the literature it was recently reported that for hypersonic reactive flow the shock stand-off distance depends not only on a reaction rate parameter, but also on the density ratio between shock and body. This is confirmed in this paper by a theoretical approach which is based on the governing conservation equations. Reasonable simplifications are introduced which for the frozen and equilibrium case allow an analytical solution for the stand-off distance on spheres. The solution method is restricted to this area since only the stand-off distance at the stagnation point is of interest. The excellent agreement achieved for the frozen or non-reactive case with well-known solutions gives evidence for the correctness of the solution method. For the equilibrium case the solution obtained shows the same behaviour as a recent study which agrees with experimental results and numerical simulations.

1. Introduction

The shock stand-off distance on simple bodies like spheres in hypersonic flow is the subject of numerous papers and the basic physical mechanisms related to this problem are quite well understood. The stand-off distance is used very often as one parameter to validate numerical methods, especially in the case of high-enthalpy reactive flows. Since for high Mach number flows on spheres the shock stand-off distance is much smaller than the body radius, its experimental determination is difficult and large errors have to be accepted. Therefore theoretical methods to determine the shock stand-off distance are of great importance not only for validation purposes but also to give more insight into the governing physical phenomena, especially for the reactive flow case. The fact that the oncoming mass flow entering the shock has to leave the flow field between shock and body is very often used to estimate the shock stand-off distance and in some sense it is part of all theoretical methods.

For the non-reactive case Van Dyke (1958) gave a numerical solution for the shock stand-off distance depending on the free-stream Mach number or density ratio across the shock. Today this solution is accepted and has been proven by many of numerical and experimental results. Extensive experiments in a ballistic range facility have been performed by Lobb (1964) who measured the shock stand-off distance on spheres by schlieren photography and compared his results with Van Dyke's numerical solution, which can be approximated quite well by (Lobb 1964)

$$\frac{\Delta_{fr}}{D} = 0.41\frac{\rho_\infty}{\rho_s}, \tag{1.1}$$

with Δ being the stand-off distance, D the sphere diameter and ρ_s the density immediately behind the shock.

For a non-equilibrium dissociating nitrogen flow Hornung (1972) calculated the shock stand-off distance on spheres and circular cylinders using two different methods. The results of both methods showed a correlation of the stand-off distance with the reaction rate parameter

$$\Omega = \left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)_s \frac{D}{2u_\infty}, \tag{1.2}$$

where $(\mathrm{d}\alpha/\mathrm{d}t)_s$ is the gradient of the dissociation fraction just behind the shock. Hornung improved the correlation by introducing the non-dimensionalized stand-off distance

$$\tilde{\Delta} = \frac{\Delta}{D}\frac{\rho_s}{\rho_\infty}, \tag{1.3}$$

which is also used in this paper. As expected, the results showed that with increasing reaction rate parameter the stand-off distance decreases, because the density behind the shock becomes larger.

Recently, Wen & Hornung (1995) extended previous results by an approximate theory which relates the dimensionless shock stand-off distance to a modified reaction rate parameter of the form

$$\tilde{\Omega}_s = \left(\frac{d\rho}{dt}\right)_s \frac{D}{\rho_s u_\infty}. \tag{1.4}$$

This approximate theory assumes linear density profiles between the shock and the body, where the density increases from its frozen value immediately behind the shock up to its value at the body. When the reaction rate is sufficiently large to achieve equilibrium between the shock and the body, the density is assumed to be constant from this point up to the body. This density profile determines an averaged value ρ_{av} which then is related to the shock stand-off distance by (Wen & Hornung 1995)

$$\tilde{\Delta} = \frac{\rho_s}{\rho_{av}} L, \tag{1.5}$$

where $L = 0.41$ is the value from (1.1) for spheres given by Van Dyke's solution. This value corresponds to the non-dimensionalized shock stand-off distance for frozen flow and has to be prescribed. It is important to note that by this approximate theory Wen & Hornung could show that the stand-off distance not only depends on the reaction rate parameter but also on the density ratio between shock and body. This was not found in the previous solutions. On the other hand the density ratio between shock and body depends on the total enthalpy of the flow. The influence of the total enthalpy was confirmed by comparison with numerical and experimental results. The excellent agreement achieved validates the inspired approach of Wen & Hornung which, in a mathematical and physical sense, is very simple.

The aim of this work is to develop a method to determine the shock stand-off distance by means as simple as possible, too, but based on the solution of the governing conservation equations. The continuity equation is integrated along radial rays $\phi = \text{const}$. The momentum equations are replaced by an approximation for the radial and tangential velocity components. The energy equation enters the problem through the fact that for steady, inviscid and adiabatic flows the total enthalpy is conserved. An ideal dissociating Lighthill gas is assumed. Recombination reactions are neglected, because the region close to the body where recombination becomes significant is a very thin layer which for overall flow field studies can be neglected (Hornung 1976). In this case the equilibrium flow definition implies a complete dissociation with $\alpha = 1$. The density jump across the shock is determined by frozen flow conditions. For equilibrium conditions with infinite reaction rate behind the shock the density jumps from its frozen value to the equilibrium one which depends on the total enthalpy of the flow. The adiabatic compression of the flow between shock and body in the stagnation region is neglected, because for high-enthalpy, high-Mach-number flow its influence is negligible on the overall flow features.

In this work only the shock stand-off distance for the stagnation streamline is of interest. Previous methods are capable of determining the whole or at least a large part of the overall blunt-body flow field which then also yields the shock stand-off distance as one part of the solution. Here, an attempt was made to reduce the mathematical effort and to find an analytical solution for the stand-off distance. Therefore, the flow field considered is restricted to an area close to the stagnation streamline which is described as function of a radial vector and the azimuthal angle ϕ. Based on this, an analytical solution is found for the shock stand-off distance on spheres for frozen and equilibrium flow. For frozen flow conditions the solution given agrees excellently with Van Dyke's solution. The deviation is only 2.4%. But for equilibrium flow the theory presented shows a strong dependence of the shock stand-off distance on the density ratio between shock and body or the total enthalpy of the flow. The shock stand-off distances determined by this theory agree well with those given by the approximate approach of Wen & Hornung. The deviation is of the order of only 2%.

2. Theoretical model for the stand-off distance

The basic equations entering the problem are the continuity and the energy equation; the first is written in polar coordinates defined in figure 1:

$$\frac{\partial}{\partial r}(\rho v r \eta) + \frac{\partial}{\partial \phi}(\rho u \eta) = 0. \tag{2.1}$$

A Lighthill ideal dissociating gas is considered (Lighthill 1957) with the specific enthalpy given by

$$h = (4 + \alpha)RT + \alpha R \Theta_d, \tag{2.2}$$

α being the dissociation fraction and Θ_d the characteristic temperature of the dissociation. It turns out that the velocity gradient in the circumferential direction is of major importance for this problem. Along the stagnation streamline its value changes approximately by a factor of 2. Just behind the shock (index s) it can be determined from the conserved tangential velocity component across the shock:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)_s = u_\infty \cos\phi. \tag{2.3}$$

At the stagnation point of the body (index b) the tangential velocity gradient is assumed to be given by the Newtonian approximation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)_b = \sqrt{\frac{2(p_b - p_\infty)}{\rho_b}}. \tag{2.4}$$

A rough estimation of the pressure at the body can be taken to be $\rho_\infty u_\infty^2$. For an incompressible flow along the stagnation streamline, i.e. $\rho_b = \rho_s$, and for air with $\rho_s/\rho_\infty = 6$ (2.3) and (2.4) yield that the velocity gradient at the shock is nearly twice that at the body. To account for this, for the simplified stagnation flow model a linear

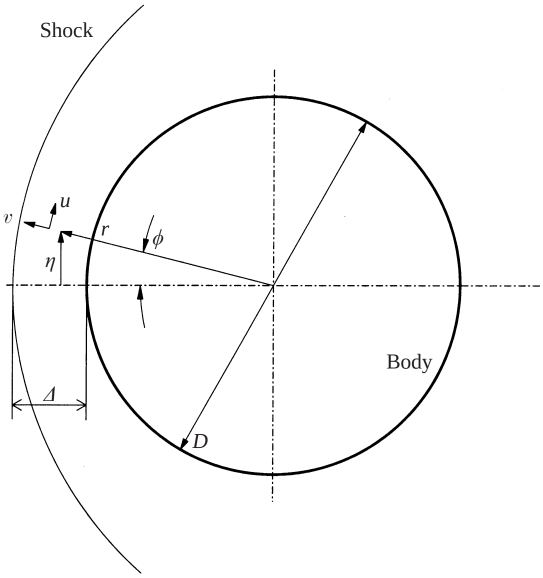

FIGURE 1. Schematic stagnation flow and notation.

distribution of the tangential velocity gradient is assumed between shock and body. The equations are non-dimensionalized by

$$\bar{p} = \frac{p}{p_b}, \quad \bar{\rho} = \frac{\rho}{\rho_b}, \quad \bar{u} = \frac{u}{u_\infty}, \quad \bar{v} = \frac{v}{u_\infty}, \quad \bar{h} = \frac{h}{h_0}, \quad \overline{T} = \frac{T}{T_{ref}},$$

$$\bar{\Theta}_d = \frac{\Theta_d}{T_{ref}}, \quad \bar{r} = \frac{2r}{D}, \quad \bar{\Delta} = \frac{2\Delta}{D},$$

with

$$h_0 = RT_{ref} = \frac{u_\infty^2}{2}.$$

Integrating the continuity equation (2.1) along a ray $\phi = \text{const}$ leads to

$$\bar{\rho}_s \bar{v}_s (1 + \bar{\Delta})^2 \sin\phi + \int_1^{1+\bar{\Delta}} \frac{\partial}{\partial \phi} (\bar{\rho}\bar{u}\bar{r} \sin\phi) \mathrm{d}\bar{r} = 0. \tag{2.5}$$

In the following only the flow region very close to the stagnation streamline is considered. For this, the following usual approximations are introduced:

$$\sin\phi \approx \phi, \quad \bar{u} = \phi \frac{\partial \bar{u}}{\partial \phi}, \quad \frac{\partial \bar{\rho}}{\partial \phi} = 0. \tag{2.6}$$

The variation of the tangential velocity gradient along the stagnation streamline is approximated by a linear profile

$$\frac{\partial \bar{u}}{\partial \phi} = \left(\frac{\partial \bar{u}}{\partial \phi}\right)_b + \left[1 - \left(\frac{\partial \bar{u}}{\partial \phi}\right)_b\right] r^*$$

with the normalized coordinate

$$r^* = \frac{\bar{r} - 1}{\bar{\Delta}}$$

ranging from 0 (body) to 1 (shock). For this the continuity equation (2.5) yields

$$\bar{\rho}_s \bar{v}_s (1+\bar{\Delta})^2 + 2\bar{\Delta} \int_0^1 \bar{\rho}(1+\bar{\Delta} r^*) \left[\left(\frac{\partial \bar{u}}{\partial \phi} \right)_b (1-r^*) + r^* \right] \mathrm{d}r^* = 0. \tag{2.7}$$

It is important to note that under the assumptions made for the stagnation region, in (2.5) the dependence on the azimuthal angle ϕ drops out. This allows a solution to be found for the stagnation streamline given in (2.7). This is the basic formula to determine the shock stand-off distance. Up to this point no gas model has been used, so that (2.7) is valid for all gas compositions. Chemical or thermal relaxation processes enter through the density distribution between shock and body and the tangential velocity gradient, which is also influenced by relaxation processes. To determine the velocity gradient at the body (2.4) is rewritten as

$$\left(\frac{\partial \bar{u}}{\partial \phi} \right)_b = \sqrt{\frac{p_b}{\rho_b h_0}}, \tag{2.8}$$

where p_∞ is neglected compared to the stagnation pressure p_b at the body. This simplification holds for all hypersonic flows.

From the equation of state

$$\frac{p_b}{\rho_b} = (1+\alpha_b) R T_b$$

and (2.2) it follows that

$$\frac{p_b}{\rho_b h_0} = \frac{1+\alpha_b}{4+\alpha_b}(1-\alpha_b \bar{\Theta}_d). \tag{2.9}$$

To make the set of equations consistent, the non-dimensionalized characteristic temperature of the dissociation $\bar{\Theta}_d$ is not free to be chosen, because it is related to the total enthalpy of the flow which has to fulfil energy conservation. To determine $\bar{\Theta}_d$ first, it is necessary to derive an expression for the density distribution between shock and body. For this the energy equation is used in the form

$$\bar{h} + \bar{u}^2 + \bar{v}^2 = 1. \tag{2.10}$$

In the neighbourhood of the stagnation streamline the term $\bar{u}^2$ is at second order small and therefore negligible. It is assumed that the radial velocity component $\bar{v}$ linearly decreases from its value $\bar{v}_s$ behind the shock to zero at the stagnation point. This, together with the prescribed tangential velocity gradient profile and (2.6), determines the two velocity components u and v, which replace the two momentum equations in the governing set of equations.

From (2.10) together with

$$\bar{v} = -\left(\frac{\rho_\infty}{\rho_s} \right) r^*$$

and the equation of state, the density distribution follows as

$$\bar{\rho} = \left(\frac{4+\alpha}{1+\alpha} \right) \left(\frac{\bar{p}}{1 - \bar{\Theta}_d \alpha - (\rho_\infty/\rho_s)^2 r^{*2}} \right) \frac{p_b}{\rho_b h_0}. \tag{2.11}$$

To solve this equation the pressure distribution has to be known. But as is known for hypersonic flow conditions the pressure variation along the stagnation streamline is not very large even for non-equilibrium flows (e.g. Hornung 1972). In this simple flow model therefore the influence of the pressure variation as well as that of the radial velocity component are neglected. The error introduced by this simplification is minimized, because in (2.11) the absence of the pressure and velocity terms affects the nominator and denominator in the same way and by roughly the same amount.

Since across the shock the flow is chemically frozen and for the free stream no dissociation is assumed, the density behind the shock is given by

$$\bar{\rho}_s = \frac{4(1+\alpha_b)(1-\alpha_b\bar{\Theta}_d)}{4+\alpha_b}.$$

For a fully equilibrium flow with $\alpha_b = 1$ this yields

$$\frac{\rho_s}{\rho_{b,e}} = \frac{8}{5}(1-\bar{\Theta}_d) \tag{2.12}$$

or

$$\bar{\Theta}_d = 1 - \frac{5}{8}\frac{\rho_s}{\rho_{b,e}}. \tag{2.13}$$

The assumptions mentioned above lead to $\bar{\rho} = 1$ for both fully frozen and equilibrium flow, which means that there is no density change between the shock and the body along the stagnation streamline. In this case (2.7) can be solved very easily which yields a quadratic equation for the stand-off distance

$$\bar{\Delta}^2\left[\bar{\rho}_s\bar{v}_s + \frac{1}{3}\left(\frac{\partial\bar{u}}{\partial\phi}\right)_b + \frac{2}{3}\right] + \bar{\Delta}\left[2\bar{\rho}_s\bar{v}_s + \left(\frac{\partial\bar{u}}{\partial\phi}\right)_b + 1\right] + \bar{\rho}_s\bar{v}_s = 0 \tag{2.14}$$

with solution

$$\tilde{\Delta} = \frac{\bar{\Delta}}{2}\frac{\rho_s}{\rho_\infty} = \left\{\frac{\rho_s}{\rho_\infty}\sqrt{\frac{1}{4}\left[1+\left(\frac{\partial\bar{u}}{\partial\phi}\right)_b\right]^2 - \frac{1}{3}\frac{\rho_s}{\rho_b}\frac{\rho_\infty}{\rho_s}\left[1+2\left(\frac{\partial\bar{u}}{\partial\phi}\right)_b\right]}\right.$$
$$\left. -\frac{1}{2}\left[1+\left(\frac{\partial\bar{u}}{\partial\phi}\right)_b\right]\frac{\rho_s}{\rho_\infty} + \frac{\rho_s}{\rho_b}\right\}\left(\frac{4}{3}+\frac{2}{3}\left(\frac{\partial\bar{u}}{\partial\phi}\right)_b - 2\frac{\rho_s}{\rho_b}\frac{\rho_\infty}{\rho_s}\right)^{-1}. \tag{2.15}$$

For frozen flow the density ratio ρ_s/ρ_b is set equal to 1. With that and for air with $\rho_\infty/\rho_s = 1/6$ equation (2.15) gives a stand-off distance of

$$\tilde{\Delta} = 0.4,$$

which agrees very well with the value $\tilde{\Delta} = 0.41$ first given by Van Dyke and which has been approved by numerous experiments (e.g. Lobb 1964). The value for the tangential velocity gradient follows from (2.8) and (2.9). The small difference of 2.4% between the two values can be attributed to the simplified stagnation flow model of this approach. It is interesting to note that for nitrogen and the frozen case Hornung (1972) determined a value for the shock stand-off distance of $\tilde{\Delta} = 0.39$ which is also slightly less than given by van Dyke's solution. For the ideal dissociating gas with $\rho_\infty/\rho_s = 1/7$ the shock stand-off distance from (2.15) is found to be $\tilde{\Delta} = 0.39$ which is smaller than for air with $\rho_\infty/\rho_s = 1/6$ due to the larger density ratio across the shock.

Equation (2.15) implies that for the non-dimensional shock stand-off distance $\tilde{\Delta}$ for the frozen or non-reactive case there is still a weak dependence on the density ratio ρ_∞/ρ_s across the shock or on the gas itself. According to (2.15), e.g. for CO_2 and $M_\infty \to \infty$ with $\rho_\infty/\rho_s = 1/7.67$, for frozen flow the non-dimensional shock stand-off distance is $\tilde{\Delta} = 0.38$, which is slightly smaller than that for air. For an atomic gas with $\gamma = 5/3$ according to (2.15) $\tilde{\Delta} = 0.44$ which is a deviation of 7.3% from the

$\rho_s/\rho_{b,e}$	0.4	0.5	0.6	0.7	0.8	0.9
$\tilde{\Delta}$ this theory (2.15)	0.164	0.203	0.241	0.280	0.319	0.359
$\tilde{\Delta}$ Wen & Hornung (2.19)	0.164	0.205	0.246	0.287	0.328	0.369

TABLE 1. Shock stand-off distance for equilibrium flow, $\rho_\infty/\rho_s = 1/6$.

value 0.41 for frozen air flow. Therefore, it is obvious that the factor L in (1.5) not only depends on the geometry but also on the gas properties.

The situation becomes more complex for equilibrium flow. It is assumed that across the shock the flow is chemically frozen (Freeman 1958). Owing to the infinite reaction rate immediately behind the shock the density jumps to the value at the body. That means that the density ρ_s just behind the shock is fixed by the free stream and frozen flow behaviour. But in this case within the shock layer the density additionally depends on the total enthalpy of the flow. This becomes obvious from (2.12), rewritten as

$$\rho_{b,e} = \frac{5}{8}\frac{h_0}{h_0 - R\Theta_d}\rho_s, \tag{2.16}$$

where $\rho_{b,e}$ is the density at the body for equilibrium flow which in this case is equal to that within the shock layer. Since for fully equilibrium flow the density within the shock layer can be varied by changing the total enthalpy, in this case the shock stand-off distance depends on the parameter $\rho_s/\rho_{b,e}$. This is obvious from (2.15), which for equilibrium flow is the same as for frozen flow but with the following substitutions:

$$\frac{\rho_s}{\rho_b} = \frac{\rho_s}{\rho_{b,e}} \tag{2.17}$$

and

$$\left(\frac{\partial \bar{u}}{\partial \phi}\right)_b = \sqrt{\frac{p_b}{\rho_b h_0}} = \frac{1}{2}\sqrt{\frac{\rho_s}{\rho_{b,e}}}. \tag{2.18}$$

The dependence of the shock stand-off distance on the density ratio $\rho_s/\rho_{b,e}$ was first derived by Wen & Hornung (1995), who assumed a linear density profile within the shock layer. From this, the average density within the shock layer is determined and is related to the shock stand-off distance. For infinite reaction rate the result of Wen & Hornung is given by

$$\tilde{\Delta} = \tilde{\Delta}_{frozen}\frac{\rho_s}{\rho_{b,e}}, \tag{2.19}$$

which directly shows that, as expected, the shock stand-off distance decreases with increasing density within the shock layer. A comparison of the shock stand-off distance calculated by (2.15) with the values achieved by Wen & Hornung is given in table 1, where in (2.19) $\tilde{\Delta}_{frozen}$ was set to the value 0.41.

The agreement achieved is excellent, the largest difference between the results of the two methods is only of the order of 2% to 3%, which on a first look is surprising, because the two approaches start from different models. Whereas Wen & Hornung assumed the density distribution and related the shock stand-off distance to the average density, the theory presented in this paper is based on the conservation equations which are solved by using suitable simplifications which have no strong influence on the basic physics of the problem. As already stated above, (2.15) shows that besides the density ratio across the shock layer the shock stand-off distance also

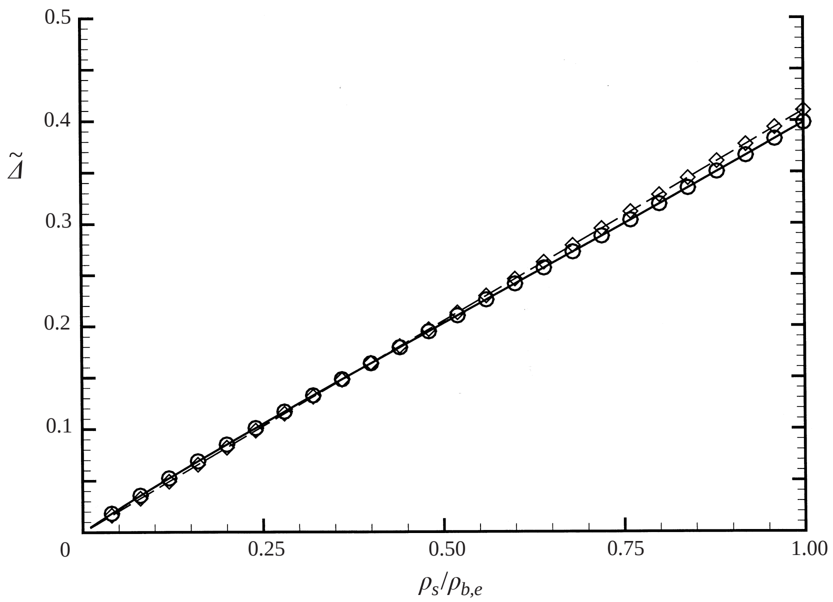

FIGURE 2. Shock stand-off distance for equilibrium flow: ○, this theory, equation (2.15), $\rho_\infty/\rho_s = 1/6$; ◇; Wen & Hornung, equation (2.19).

depends on the tangential velocity gradient, because this determines the mass flow rate out off the stagnation region. It is interesting to note that according to (2.18) for equilibrium flow the tangential velocity gradient depends on the density ratio $\rho_s/\rho_{b,e}$, i.e. both the density and velocity field are influenced by high-temperature effects in the stagnation region. This shows that to determine the shock stand-off distance the density as well as the velocity field have to be taken into account. In the approach described by (2.19) the dependence of the velocity field is taken into account by the constant factor $\tilde{\Delta}_{frozen} = L = 0.41$, because this value results from numerical simulations of a non-reacting flow field around a sphere. It does not take into account the influence of high-temperature effects on the velocity field, as given by (2.18) of this theory, nor the dependence of the non-dimensionalized shock stand-off distance $\tilde{\Delta}$ on different gas compositions, even for the frozen case. This also shows the different approaches of the two methods.

In figure 2 the non-dimensional stand-off distance is presented as function of the density ratio $\rho_s/\rho_{b,e}$ for equilibrium flow. It is obvious that the linear dependence in (2.19) agrees quite well with the result of the theory presented in this paper. In figure 3, taken from Wen & Hornung, the limiting values for frozen and equilibrium flow obtained by this theory are indicated.

3. Conclusions

A theoretical model has been developed to determine the shock stand-off distance on axisymmetric blunt bodies for frozen and equilibrium flows. No use is made of previous results like Van Dyke's solution for blunt-body flows. The restriction of the flow field considered to an area close to the axis and the use of suitable simplifications allow an analytical solution for the shock stand-off distance. The solution for the frozen case agrees very well with the well-known solution of Van Dyke. For larger reaction rates and the limiting case of equilibrium flow Wen & Hornung's result,

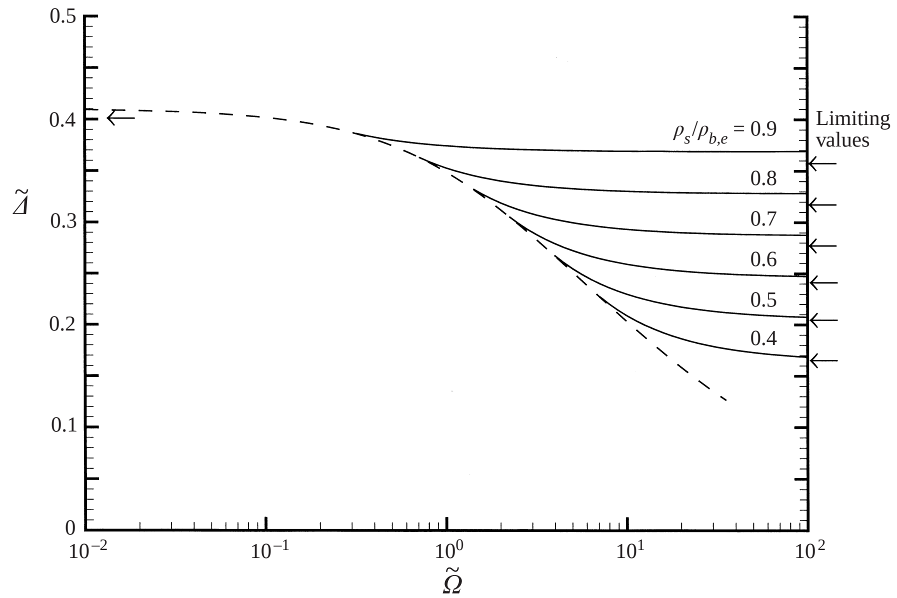

FIGURE 3. Shock stand-off distance as function of reaction rate parameter (Wen & Hornung) and limiting values for frozen and equilibrium flow according to this theory, equation (2.15), $\rho_\infty/\rho_s = 1/6$.

showing the dependence of the shock stand-off distance on the density ratio between shock and body, is confirmed and very good agreement is achieved. It becomes obvious in the theory presented that in addition to the density the tangential velocity gradient is also of great importance for the stand-off distance, since like the density it determines the mass flow rate leaving the stagnation region. The theory shows that the influence of the velocity field on the stand-off distance depends on the density ratio between shock and body. For equilibrium flow the magnitude of this ratio is determined by the total enthalpy of the flow.

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REFERENCES

FREEMAN, N. C. 1958 Non-equilibrium flow of an ideal dissociating gas. *J. Fluid Mech.* **4**, 407–425.

HORNUNG, H. G. 1972 Non-equilibrium dissociating flow over spheres and circular cylinders. *J. Fluid Mech.* **53**, 149–176.

HORNUNG, H. G. 1976 Non-equilibrium ideal-gas dissociation after a curved shock wave. *J. Fluid Mech.* **74**, 143–159.

LIGHTHILL, M. J. 1957 Dynamics of a dissociating gas. Part I. Equilibrium flow. *J. Fluid Mech.* **2**, 1–32.

LOBB, R. K. 1964 Experimental measurement of shock detachment distance on spheres fired in air at hypervelocities. In *The High Temperature Aspects of Hypersonic Flow* (ed. W. C. Nelson), pp. 519–527. Pergamon.

VAN DYKE, M. D. 1958 The supersonic blunt-body problem – review and extension. *J. Aero. Sci.* **25**,485–496.

WEN, C. Y. & HORNUNG, H. G. 1995 Non-equilibrium dissociating flow over spheres. *J. Fluid Mech.* **299**, 389–405.